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On Deficient Cubic Spline Interpolation

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1. INTRODUCTION

Following Schoenberg [5] and de Boor [2] one of the present authors [3] has treated cubic spline interpolation by matching the integral mean of a function and a spline between successive equidistant knots. Earlier Sharma and Tzimbalario [6] had studied quadratic splines with similar matching conditions. Our object is to study deficient cubic splines by making less restrictive continuity requirements at the joints and having two interpolatory conditions, one of which is the matching condition at appropriate points of the dividing intervals while the other is the matching of the integral means. Corresponding error bound is also obtained in the present paper. It may be mentioned here that sharp error bounds in different norms have been given by Varga in [7; Theorem 2.1], where spline interpolation of bounded linear functionals is studied.

2. REPRESENTATION OF CUBIC SPLINE

Let Δ: $0 = x_0 < x_1 \cdots < x_n = 1$ denote a partition of [0, 1] with equidistant knots x_i so that $h = x_i - x_{i-1} = 1/n$. Let π_l be the set of all real algebraic polynomials of degree at most l. We define the deficient polynomial spline class $S(l, \Delta)$ as

$$S(l, \Delta) = \{s(x) \mid s(x) \in C^{l-2}[0, 1], s(x) \in \pi_l, \\ x \in [x_{i-1}, x_i], i = 1, 2, \ldots, n\}.$$

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Consider a function $g(x)$ defined in [0, 1] such that

$$g(x + h) - g(x) = \text{a constant } K', \tag{2.1}$$

for example.

Along with (2.1) we also assume that $g(x)$ satisfies one of the following conditions:

$$g(x) = K \qquad (0 \leqslant x < mh;\ 2/9 \leqslant m < 1/2), \tag{2.2a}$$

$$g(x) = K \qquad (mh \leqslant x \leqslant h;\ 1/2 \leqslant m \leqslant 7/9), \tag{2.2b}$$

where K is any constant. Writing $\theta_i = x_{i-1} + mh$; we propose:

PROBLEM A. *Let f be a* 1*-periodic locally integrable function with respect to a positive measure dg, where g satisfies conditions* (2.1) *and* (2.2a) (*or* (2.2b)). *Find* 1*-periodic* $s(x) \in S(3, \Delta)$ *satisfying the conditions*:

$$f(\theta_i) = s(\theta_i), \qquad \int_{x_{i-1}}^{x_i} (f - s)(x)\, dg = 0, \qquad i = 1, 2,..., n. \tag{2.3}$$

Choosing $g(x)$ such that in $[0, h]$

$$\begin{aligned} g(x) &= K \qquad \text{for } 0 \leqslant x \leqslant m'h \\ &= K + 1 \qquad \text{for } m'h < x \leqslant h, \end{aligned}$$

we may extend the definition of $g(x)$ over [0, 1] by assuming condition (2.1). In this case the area matching condition in (2.3) reduces to the interpolatory condition at the points $\theta_i' = x_{i-1} + m'h$, $i = 1, 2,..., n$. If we now take $m' > m$, then (2.3) gives the interpolating conditions at the points θ_i and θ_i'. Thus it follows from a known result ([4], p. 246) that Problem A, with the above choice of g has a unique solution even for non-equidistant knots provided $\frac{1}{2} < m + m' < \frac{3}{2}$ and $0 \leqslant m < m' \leqslant 1$. It may be observed that the matching condition (2.3) of Problem A has the scope of providing interpolatory conditions for a wider choice of m and m'. For example, if we take $m < m'$, $2/9 \leqslant m$ and $m + m' \leqslant \frac{1}{2}$, then the hypotheses of Problem A are satisfied for the above choice of g and we obtain interpolatory conditions, which are not covered by the result proved in [4].

Writing $s'(x_i) = M_i$, $i = 0, 1,..., n$, we now proceed to obtain a representation of cubic spline $s(x)$ of our Problem A in terms of M_i's. Since $s'(x)$ is a quadratic in $x_{i-1} \leqslant x \leqslant x_i$, we have for $x \in [x_{i-1}, x_i]$

$$\begin{aligned} h^2 s'(x) = {} & M_i[(x - x_{i-1}) - 2(x_i - x)](x - x_{i-1}) \\ & + M_{i-1}[(x_i - x) - 2(x - x_{i-1})](x_i - x) \\ & + 6\beta_i(x_i - x)(x - x_{i-1}), \end{aligned} \tag{2.4}$$

where β_i's are constants to be determined by the hypotheses concerning $s(x)$. Integrating (2.4) and simplifying, we get

$$\begin{aligned} h^2 s(x) = & -M_i(x - x_{i-1})^2(x_i - x) \\ & -M_{i-1}[\tfrac{1}{3}(x_i - x)^3 + \tfrac{1}{3}(x - x_{i-1})^3 + (x - x_{i-1})^2(x_i - x)] \\ & + \beta_i(x - x_{i-1})^2[3(x_i - x) + (x - x_{i-1})] + \delta_i h^2 \end{aligned} \tag{2.5}$$

and using the continuity of $s(x)$ at x_i, we have

$$\beta_i h = \tfrac{1}{3}h(M_{i-1} - M_i) + \delta_{i+1} - \delta_i . \tag{2.6}$$

For $i = 1, 2, \dots, n$, we set

$$\int_0^h x^r(h - x)^p \, dg = h^{r+p}K(r, p), \qquad r, p = 0, 1, 2, 3;$$

and observe that in view of condition (2.1),

$$\int_{x_{i-1}}^{x_i} (x - x_{i-1})^r (x_i - x)^p \, dg = h^{r+p}K(r, p),$$

for all r, p and i.

3. Existence and Uniqueness

We now answer Problem A in the following:

THEOREM 1. *There exists a unique* $s(x) \in S(3, \Delta)$ *which satisfies the conditions of Problem A provided that* $g(x)$ *does not have just one jump in* $[0, h]$ *at the point* mh, $0 \leqslant m \leqslant 1$.

For the proof of our Theorem we need the following.

LEMMA. *Suppose* $g(x)$ *does not have just one jump at* mh, $0 \leqslant m \leqslant 1$ *in* $[0, h]$ *and let*

$$\begin{aligned} K_1(m) &= h^{-3} \int_0^h (x - mh)\, \alpha(m, x)\, dg, \\ K_2(m) &= h^{-3} \int_0^h (x - mh)[\alpha(m, x) - h^2]\, dg, \end{aligned} \tag{3.1}$$

where $\alpha(m, x) = -2x^2 + h(3 - 2m)(x + mh)$. *Then both* $K_1(m)$, $K_2(m)$ *are positive or negative according as* (2.2a) *holds or* (2.2b) *holds.*

Proof of the Lemma. We see that $\alpha'(m, x)$ vanishes at only one point in $[0, h]$ and $\alpha(m, x) > h^2$ at that point provided $2/9 \leqslant m \leqslant 7/9$. Also $\alpha(m, x) > h^2$ at $x = 0, h$. Now, if we assume (2.2a), then

$$K_2(m) = h^{-3} \int_{mh}^{h} (x - mh)[\alpha(m, x) - h^2]\, dg > 0,$$

since the integrand is positive for $x > mh$ and $g(x)$ does not have just one jump at mh. This, of course, gives that $K_1(m) > 0$. The other part of the lemma follows by similar reasoning.

Proof of Theorem 1.

In order to eliminate δ_i, δ_{i+1} and β_i from Eqs. (2.5) and (2.6), we observe that in view of the first matching condition in (2.3), we have

$$f(\theta_i) = -2M_i m^2 t_2 + M_{i-1} h(m(1 - m)^2 - \tfrac{1}{3}) + \beta_i h t_4 + \delta_i, \quad (3.2)$$

where $t_2 = \frac{1}{2}h(1 - m)$ and $t_4 = m^2(3 - 2m)$. Comparing the values of $\beta_i h$ as obtained from (2.6) and (3.2), we get

$$\begin{aligned} f(\theta_i) + \tfrac{1}{3}M_i h m^2(6 - 5m) - \tfrac{1}{3}M_{i-1}h(m - 1)^3 \\ = t_4(\delta_{i+1} - \delta_i) + \delta_i . \end{aligned} \quad (3.3)$$

Now eliminating β_i between (2.5) and (3.2), we have

$$\begin{aligned} h^3 t_4 s(x) = {} & M_i h m^2[(1 - m)(x - x_{i-1})^3 - m(x - x_{i-1})^2(x_i - x)] \\ & -\tfrac{1}{3}hM_{i-1}[(m - 1)^3\{3(x - x_{i-1})^2(x_i - x) + (x - x_{i-1})^3\} \\ & + t_4(x_i - x)^3] \\ & + (f(\theta_i) - \delta_i)(x - x_{i-1})^2[3(x_i - x) + (x - x_{i-1})] \\ & + \delta_i t_4 h^3. \end{aligned} \quad (3.4)$$

Thus, using the area matching condition in (2.3) and writing $F_i = \int_{x_{i-1}}^{x_i} f\, dg$, we have

$$\begin{aligned} & M_i h m^2[(1 - m)K(3, 0) - mK(2, 1)] \\ & \quad -\tfrac{1}{3}M_{i-1}h[(m - 1)^3\{3K(2, 1) + K(3, 0)\} + t_4 K(0, 3)] \\ & \quad + (f(\theta_i) - \delta_i)[3K(2, 1) + K(3, 0)] + \delta_i t_4 K(0, 0) = F_i t_4 . \end{aligned} \quad (3.5)$$

We see that the coefficient of $-\delta_i$ in (3.5) is $K_1(m)$ which is nonzero by virtue of the lemma. Thus, we may determine δ_i, δ_{i+1} from the equations of the type (3.5). Substituting the values so obtained in (3.3) and observing that

$$\begin{aligned} K(2, 1) &= K(2, 0) - K(3, 0), \\ K(0, 3) &= K(0, 0) - 3K(1, 0) + 3K(2, 0) - K(3, 0), \end{aligned}$$

we have, after some simplification a linear system of equations in M_i's $(i = 1, 2,..., n)$:

$$p(m)\, M_{i+1} + q(m)\, M_i + r(m)\, M_{i-1} = u_i \tag{3.6}$$

where $p(m) = hm^2[K(3, 0) - mK(2, 0)]$,

$$\begin{aligned} q(m) &= h[(1 - m)\, m^2 K(0, 0) + t_4 K(1, 0) \\ &\quad + (2m^2 - 2m - 1)\{(m + 1)K(2, 0) - K(3, 0)\}], \\ r(m) &= h(1 - m)^2[-mK(0, 0) + (2m + 1)K(1, 0) \\ &\quad - (m + 2)K(2, 0) + K(3, 0)], \\ u_i &= (1 - t_4)\, F_i + t_4 F_{i+1} - f(\theta_i)K(0, 0) \\ &\quad + t_6(f(\theta_i) - f(\theta_{i+1})) \end{aligned}$$

and $t_6 = 3K(2, 0) - 2K(3, 0)$.

Considering first the case in which $g(x) = K$ for $x \in [0, mh]$, we notice that the coefficients of M_{i+1} and M_{i-1} are positive whereas the coefficient of M_i is negative. Further, the excess of the positive value of the coefficient of M_i over the sum of the coefficients of M_{i-1} and M_{i+1} is $K_2(m)$ which is positive by virtue of the lemma. Thus, Eqs. (3.6) exhibit the diagonal dominant property and have a unique system of solutions.

In the other case in which $g(x) = K$ for $x \in [mh, h]$, the coefficient of M_i is positive whereas the coefficients of M_{i+1} and M_{i-1} are negative and the sum of the coefficients is $-K_2(m)$ which is positive by virtue of the lemma. Thus, Eqs. (3.6) have the diagonal dominant property in this case also. This completes the proof of Theorem 1.

4. Error Bounds

We shall consider in this section error bounds for the spline of Theorem 1. Let us write Eqs. (3.6) as

$$A_m M = U_m, \tag{4.1}$$

where A_m is the coefficient matrix and M and U_m are single column matrices (M_i) and (u_i), respectively. It may be observed that (cf. [1], p. 21) the row-max norm:

$$\| A_m^{-1} \| \leqslant \{| q(m)| - | p(m)| - | r(m)|\}^{-1} = | K_2(m)|^{-1}, \tag{4.2}$$

where $K_2(m)$ is given by (3.1).

Since under the hypothesis (2.1),

$$F_{i+i} - F_i = \int_{x_i}^{x_{i+1}} (f - f_i)\, dg + \int_{x_{i-1}}^{x_i} (f_i - f)\, dg,$$

therefore, we have $\| U_m \| \leqslant K_3(m)w(f; h)$, where w is the modulus of continuity of $f(x)$ and

$$K_3(m) = [(2t_4 + 1)K(0, 0) + 6K(2, 1) + 2K(3, 0)]. \tag{4.3}$$

Thus, we have from (4.2)

$$\| M \| \leqslant K_3(m) \mid K_2(m)|^{-1}w(f; h). \tag{4.4}$$

If F denotes the single column matrix (f_i') then we have from (4.1)

$$A_m(M - F) = (U_m - A_mF). \tag{4.5}$$

Now using the law of the mean, we get

$$\begin{aligned} \| U_m - A_mF \| = \max_i \mid & hf'(\eta_i)\{t_4K(0, 0) + (1 - t_4) K(1, 0) - t_6\} \\ & + hf'(\eta_{i+1}) t_4K(1, 0) + mhf'(\xi_i)\{t_6 - K(0, 0)\} \\ & - mhf'(\xi_{i+1}) t_6 - r(m) f_{i+1}' - q(m) f_i' - p(m)f_{i-1}' \mid, \end{aligned} \tag{4.6}$$

where ξ_i, η_i are points interior to $[x_{i-1}, x_i]$ such that $\xi_i < \theta_i$. Thus, rearranging the terms in (4.6) suitably, we have

$$\| U_m - A_mF \| \leqslant K_4(m) w_1(f; h), \tag{4.7}$$

where $w_1(f; h)$ is the modulus of continuity of $f'(x)$ and

$$\begin{aligned} K_4(m) = & (-3m^3 + 4m^2 + m)K(0, 0) + K(1, 0) + (2m^3 - 3m + 8)K(2, 0) \\ & + (2m^2 - 2m + 5)K(3, 0). \end{aligned}$$

Thus, in view of (4.2) and (4.5), we have

$$\| M - F \| \leqslant | K_2(m)|^{-1}K_4(m) w_1(f; h). \tag{4.8}$$

We are now set to prove the following.

THEOREM 2. *Suppose $s(x)$ is the deficient cubic spline of Theorem 1 interpolating $f(x)$ and $f(x) \in C^1[0, 1]$. Then*

$$\max_x | s(x) - f(x)| \leqslant K(m) w(f; h) \tag{4.9}$$

and

$$\max_x | s'(x) - f'(x)| \leqslant K'(m) w(f; h) + K''(m) w_1(f; h), \tag{4.10}$$

where $K(m)$, $K'(m)$ and $K''(m)$ are functions only depending on m.

Proof of Theorem 2.

In order to get the error bounds for $s(x) - f(x)$ we just consider the case in which (2.2a) holds. Similar considerations give the bounds for the other case. Substituting the value of δ_i from (3.5) in (3.4) and writing $x = \frac{1}{2}(x_{i-1} + x_i + mh) + \epsilon$, we have

$$\begin{aligned}
&h^3 t_4(s(x) - f(x)) \\
&\quad = M_i h m^2[(t_1 + \epsilon)^2(t_2 + \epsilon) \\
&\qquad -\{K_1(m)\}^{-1}\{t_4 h^3 - (t_1 + \epsilon)^2(t_3 - 2\epsilon)^2\}(K(3, 0) \\
&\qquad - mK(2, 0))] \\
&\qquad -\tfrac{1}{3} h M_{i-1}[(m - 1)^3(t_3 - 2\epsilon)(t_1 + \epsilon)^2 + t_4(t_2 - \epsilon)^3 \\
&\qquad - \{K_1(m)\}^{-1}\{t_4 h^3 - (t_2 + \epsilon)^2\}(t_3 - 2\epsilon)\{(m - 1)^3(3K(2, 1) \\
&\qquad + K(3, 0)) + t_4 K(0, 3)\}] \\
&\qquad + \{K_1(m)\}^{-1} t_4[t_4 h^2 - (t_1 + \epsilon)^2(t_3 - 2\epsilon)][F_i - f(\theta_i)K(0, 0)] \\
&\qquad + t_4 h^2(f(\theta_i) - f(x)),
\end{aligned}$$

where $t_1 = \frac{1}{2}h(1 + m)$, and $t_3 = (2 - m)h$. Now observing that $|\epsilon| \leqslant t_1$ and applying (4.4) we get

$$\max_x |s(x) - f(x)| \leqslant K(m)\, w(f; h),$$

where $K(m)$ depends only on m.

Next differentiating (3.3) and substituting the value of δ_i from (3.4) and setting $x = \frac{1}{2}(x_{i-1} + x_i + mh) + \epsilon$ we get

$$\begin{aligned}
&h^2 t_4(s'(x) - M_i) \\
&\quad = M_i h[m^2(t_1 + \epsilon)\{\tfrac{1}{2}(h + t_3) + 3\epsilon\} - h^2 t_4 \\
&\qquad + 6(K_1(m))^{-1} t_5(K(3, 0) - mK(2, 0))] \\
&\qquad -6(K_1(m))^{-1} t_4 t_5[F_i - f(\theta_i)K(0, 0)] \\
&\qquad -\tfrac{1}{3} h M_{i-1}[-3t_4(t_2 - \epsilon)^2 \\
&\qquad +6(K_1(m))^{-1} t_4 t_5((m - 1)^3 K(0, 0) + K(0, 3))],
\end{aligned}$$

where $t_5 = \frac{1}{4}h^2 - (\frac{1}{2}mh + \epsilon)^2$. Thus, observing that $|\epsilon| \leqslant t_1$ and applying (4.4) we have

$$\max_x |s'(x) - M_i| \leqslant K_5(m)\, w(f; h), \tag{4.11}$$

where $K_5(m)$ depends only on m.

Now observing that

$$|s'(x) - f'(x)| \leqslant |s'(x) - M_i| + |M_i - f_i'| + |f_i' - f'(x)|$$

we obtain (4.10) when we appeal to (4.8) and (4.11).

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References

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, "The Theory of Splines and Their Applications," Academic Press, New York, 1967.
2. C. de Boor, Appendix to "Splines and Histograms" pp. 329–358, ISNM, Vol. 21, Birkhäuser Verlag, Basel.
3. H. P. Dikshit, On Cubic spline interpolation, *J. Approx. Theory* **22** (1978), 105–111.
4. A. Meir and A. Sharma, Convergence of a class of interpolatory splines, *J. Approx. Theory* **1** (1968), 243–250.
5. I. J. Schoenberg, "Splines and Histograms (Splines and Approximation Theory)," pp. 277–327, ISNM, Vol. 21, Birkhäuser Verlag, Basel.
6. A. Sharma and J. Tzimbalario, Quadratic Splines, *J. Approx. Theory* **19** (1977), 186–193.
7. R. S. Varga, Error bounds for spline interpolation, In "Approximation with Special Emphasis on Spline Functions" (I. J. Schoenberg, Ed.), pp. 367–388, Academic Press, New York, 1969.